Topological Factors Derived From Bohmian Mechanics

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Abstract

We derive for Bohmian mechanics topological factors for quantum systems with a multiply-connected configuration space $\mathcal{Q}$. These include nonabelian factors corresponding to what we call holonomy-twisted representations of the fundamental group of $\mathcal{Q}$. We employ wave functions on the universal covering space of $\mathcal{Q}$. As a byproduct of our analysis, we obtain an explanation, within the framework of Bohmian mechanics, of the fact that the wave function of a system of identical particles is either symmetric or anti-symmetric.

Key words: topological phases, multiply-connected configuration spaces, Bohmian mechanics, universal covering space
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Dedicated to Rafael Sorkin on the occasion of his 60th birthday

1 Introduction

We consider here a novel approach towards topological effects in quantum mechanics. These effects arise when the configuration space $\mathcal{Q}$ of a quantum system is a

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multiply-connected Riemannian manifold and involve *topological factors* forming a representation (or holonomy-twisted representation) of the fundamental group $\pi_1(\mathcal{Q})$ of $\mathcal{Q}$. Our approach is based on Bohmian mechanics [5, 2, 8, 3, 9, 12], a version of quantum mechanics with particle trajectories. The use of Bohmian paths allows a derivation of the link between homotopy and quantum mechanics that is essentially different from derivations based on path integrals.

The topological factors we derive are equally relevant and applicable in orthodox quantum mechanics, or any other version of quantum mechanics. Bohmian mechanics, however, provides a sharp mathematical justification of the dynamics with these topological factors that is absent in the orthodox framework. Different topological factors give rise to different Bohmian dynamics, and thus to different quantum theories, for the same configuration space $\mathcal{Q}$ (whose metric we regard as incorporating the "masses of the particles"), the same potential, and the same value space of the wave function.

The motion of the configuration in a Bohmian system of N distinguishable particles can be regarded as corresponding to a dynamical system in the configuration space $\mathcal{Q} = \mathbb{R}^{3N}$, defined by a time-dependent vector field v^{ψ_t} on $\mathcal{Q}$ which in turn is defined, by the Bohmian law of motion, in terms of ψ_t. We are concerned here with the analogues of the Bohmian law of motion when $\mathcal{Q}$ is, instead of $\mathbb{R}^{3N}$, an arbitrary Riemannian manifold.[1] The main result is that, if $\mathcal{Q}$ is multiply connected, there are several such analogues: several dynamics, which we will describe in detail, corresponding to different choices of the topological factors.

It is easy to overlook the multitude of dynamics by focusing too much on just one, the simplest one, which we will define in Section 2: the *immediate generalization* of the Bohmian dynamics from $\mathbb{R}^{3N}$ to a Riemannian manifold, or, as we shall briefly call it, the *immediate Bohmian dynamics*. Of the other kinds of Bohmian dynamics, the simplest involve phase factors associated with non-contractible loops in $\mathcal{Q}$, forming a character[2] of the fundamental group $\pi_1(\mathcal{Q})$. In other cases, the topological factors are given by matrices or endomorphisms, forming a unitary representation of $\pi_1(\mathcal{Q})$ or, in the case of a vector bundle, a holonomy-twisted representation (see the end of Section 4 for the definition). As we shall explain, the dynamics of bosons is an "immediate" one, but not the dynamics of fermions (except when using a certain not entirely natural vector bundle). The Aharonov–Bohm effect can be regarded as an example of a non-immediate dynamics on the accessible region of 3-space.

It is not obvious what "other kinds of Bohmian dynamics" should mean. We will investigate one approach here, while others will be studied in forthcoming works. The present approach is based on considering wave functions ψ that are defined not on the configuration space $\mathcal{Q}$ but on its universal covering space $\widehat{\mathcal{Q}}$. We then investigate which kinds of periodicity conditions, relating the values on different levels of the

[1] Manifolds will throughout be assumed to be Hausdorff, paracompact, connected, and C^∞. They need not be orientable.

[2] By a *character* of a group we refer to what is sometimes called a unitary multiplicative character, i.e., a one- dimensional unitary representation of the group.

covering fiber by a topological factor, will ensure that the Bohmian velocity vector field associated with ψ is projectable from $\widehat{\mathcal{Q}}$ to $\mathcal{Q}$. This is carried out in Section 3 for scalar wave functions and in Section 4 for wave functions with values in a complex vector space (such as a spin-space) or a complex vector bundle. In the case of vector bundles, we derive a novel kind of topological factor, given by a holonomy-twisted representation of $\pi_1(\mathcal{Q})$.

The notion that multiply-connected spaces give rise to different topological factors is not new. The most common approach is based on path integrals and began largely with the work of Schulman [18, 19] and Laidlaw and DeWitt [14]; see [20] for details. Nelson [17] derives the topological phase factors for scalar wave functions from stochastic mechanics. There is also the current algebra approach of Goldin, Menikoff, and Sharp [11].

2 Bohmian Mechanics in Riemannian manifolds

Bohmian mechanics can be formulated by appealing only to the Riemannian structure g of the configuration $\mathcal{Q}$ space of a physical system: the state of the system in Bohmian mechanics is given by the pair (Q, ψ); $Q \in \mathcal{Q}$ is the configuration of the system and ψ is a (standard quantum mechanical) wave function on the configuration space $\mathcal{Q}$, taking values in some *Hermitian vector space* W, i.e., a finite-dimensional complex vector space endowed with a positive-definite Hermitian (i.e., conjugate-symmetric and sesqui-linear) inner product $(\,\cdot\,,\,\cdot\,)$.

The state of the system changes according to the guiding equation and Schrödinger's equation [8]:

$$\frac{dQ_t}{dt} = v^{\psi_t}(Q_t) \tag{1a}$$

$$i\hbar\frac{\partial\psi_t}{\partial t} = -\tfrac{\hbar^2}{2}\Delta\psi_t + V\psi_t\,, \tag{1b}$$

where the Bohmian velocity vector field v^ψ associated with the wave function ψ is

$$v^\psi := \hbar\,\mathrm{Im}\frac{(\psi, \nabla\psi)}{(\psi,\psi)}. \tag{2}$$

In the above equations Δ and ∇ are, respectively, the Laplace-Beltrami operator and the gradient on the configuration space equipped with this Riemannian structure; V is the potential function with values given by Hermitian matrices (endomorphisms of W). Thus, given $\mathcal{Q}$, W, and V, we have specified a Bohmian dynamics, the *immediate Bohmian dynamics*.[3]

[3] Since the law of motion involves a derivative of ψ, the merely measurable functions in $L^2(\mathcal{Q})$ will of course not be adequate for defining trajectories. However, we will leave aside the question, from which dense subspace of $L^2(\mathcal{Q})$ should one choose ψ. For a discussion of the global existence question of Bohmian trajectories in $\mathbb{R}^{3N}$, see [4, 21].

The empirical agreement between Bohmian mechanics and standard quantum mechanics is grounded in equivariance [8, 10]. In Bohmian mechanics, if the configuration is initially random and distributed according to $|\psi_0|^2$, then the evolution is such that the configuration at time t will be distributed according to $|\psi_t|^2$. This property is called the equivariance of the $|\psi|^2$ distribution. It follows from comparing the transport equation arising from (1a)

$$\frac{\partial \rho_t}{\partial t} = -\nabla \cdot (\rho_t v^{\psi_t}) \tag{3}$$

for the distribution ρ_t of the configuration Q_t, where $v^\psi = (v_1^\psi, \ldots, v_N^\psi)$, to the quantum continuity equation

$$\frac{\partial |\psi_t|^2}{\partial t} = -\nabla \cdot (|\psi_t|^2 v^{\psi_t}), \tag{4}$$

which is a consequence of Schrödinger's equation (1b). A rigorous proof of equivariance requires showing that almost all (with respect to the $|\psi|^2$ distribution) solutions of (1a) exist for all times. This was done in [4, 21]. A more comprehensive introduction to Bohmian mechanics may be found in [12, 3, 9].

An important example (with, say, $W = \mathbb{C}$) is that of several particles moving in a Riemannian manifold M, a possibly curved physical space. Then the configuration space for N distinguishable particles is $\mathcal{Q} := M^N$. Let the masses of the particles be m_i and the metric of M be g. Then the relevant metric on M^N is

$$g^N(v_1 \oplus \cdots \oplus v_N, w_1 \oplus \cdots \oplus w_N) := \sum_{i=1}^N m_i g(v_i, w_i).$$

Using g^N allows us to write (2) and (1a) instead of the equivalent equations

$$\frac{d\boldsymbol{Q}_k}{dt} = \frac{\hbar}{m_k} \mathrm{Im} \frac{(\psi, \nabla_k \psi)}{(\psi, \psi)}(\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_N), \quad k = 1, \ldots, N \tag{5}$$

$$i\hbar \frac{\partial \psi}{\partial t} = -\sum_{k=1}^N \frac{\hbar^2}{2m_k} \Delta_k \psi + V\psi, \tag{6}$$

where $\boldsymbol{Q}_k$, the k^{th} component of Q, lies in M, and ∇_k and Δ_k are the gradient and the Laplacian with respect to g, acting on the k^{th} factor of M^N. Another important example [14] is that of N identical particles in $\mathbb{R}^3$, for which the natural configuration space is the set ${}^N\mathbb{R}^3$ of all N-element subsets of $\mathbb{R}^3$,

$${}^N\mathbb{R}^3 := \left\{ S | S \subseteq \mathbb{R}^3, |S| = N \right\}, \tag{7}$$

which inherits a Riemannian metric from $\mathbb{R}^3$. Spin is incorporated by choosing for W a suitable spin space [2]. For one particle moving in $\mathbb{R}^3$, we may take W to be a complex, irreducible representation space of $SU(2)$, the universal covering group[4] of the rotation group $SO(3)$. If it is the spin-s representation then $W = \mathbb{C}^{2s+1}$.

More generally, we can consider a Bohmian dynamics for wave functions taking values in a complex vector bundle E over the Riemannian manifold $\mathcal{Q}$. That is, the value space then depends on the configuration, and wave functions become sections of the vector bundle. Such a case occurs for identical particles with spin s, where the bundle E of spin spaces over the configuration space $\mathcal{Q} = {}^N\mathbb{R}^3$ consists of the $(2s+1)^N$-dimensional spaces

$$E_q = \bigotimes_{\boldsymbol{q} \in q} \mathbb{C}^{2s+1}, \quad q \in \mathcal{Q}. \tag{8}$$

For a detailed discussion of this bundle, of why this is the right bundle, and of the notion of a tensor product over an arbitrary index set, see [7].

We introduce now some notation and terminology.

Definition 1. *A* Hermitian vector bundle*, or* Hermitian bundle*, over $\mathcal{Q}$ is a finite-dimensional complex vector bundle E over $\mathcal{Q}$ with a connection and a positive-definite, Hermitian local inner product $(\,\cdot\,,\,\cdot\,) = (\,\cdot\,,\,\cdot\,)_q$ on E_q, the fiber of E over $q \in \mathcal{Q}$, which is parallel.*

Our bundle, the one of which ψ is a section, will always be a Hermitian bundle. Note that since a Hermitian bundle consists of a vector bundle and a connection, it can be nontrivial even if the vector bundle is trivial: namely, if the connection is nontrivial. The *trivial Hermitian bundle* $\mathcal{Q} \times W$, in contrast, consists of the trivial vector bundle with the trivial connection, whose parallel transport P_β, in general a unitary endomorphism from E_q to $E_{q'}$ for β a path from q to q', is always the identity on W. The case of a W-valued function $\psi : \mathcal{Q} \to W$ corresponds to the trivial Hermitian bundle $\mathcal{Q} \times W$.

The global inner product on the Hilbert space of wave functions is the local inner product integrated against the Riemannian volume measure associated with the metric g of $\mathcal{Q}$,

$$\langle \phi, \psi \rangle = \int_{\mathcal{Q}} dq\, (\phi(q), \psi(q))\,.$$

The Hilbert space equipped with this inner product, denoted $L^2(\mathcal{Q}, E)$, contains the square-integrable, measurable (not necessarily smooth) sections of E modulo equality almost everywhere. The covariant derivative $D\psi$ of a section ψ is an "E-valued 1-form," i.e., a section of $\mathbb{C}T\mathcal{Q}^* \otimes E$ (with $T\mathcal{Q}^*$ the cotangent bundle), while we write $\nabla\psi$ for the section of $\mathbb{C}T\mathcal{Q} \otimes E$ metrically equivalent to $D\psi$. The potential V is now a self-adjoint section of the endomorphism bundle $E \otimes E^*$ acting on the vector

[4] The universal covering space of a Lie group is again a Lie group, the *universal covering group*. It should be distinguished from another group also called the *covering group*: the group $Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$ of the covering (or deck) transformations of the universal covering space $\widehat{\mathcal{Q}}$ of a manifold $\mathcal{Q}$, which will play an important role later.

bundle's fibers. The equations defining the Bohmian dynamics are, *mutatis mutandis*, the same equations (1) and (2) as before.

We wish to introduce now further Bohmian dynamics beyond the immediate one. To this end, we will consider wave functions on $\widehat{\mathcal{Q}}$, the universal covering space of $\mathcal{Q}$. This idea is rather standard in the literature on quantum mechanics in multiply-connected spaces [14, 6, 15, 16, 13]. However, the complete specification of the possibilities that we give in Section 4 includes some, corresponding to what we call *holonomy-twisted representations* of $\pi_1(\mathcal{Q})$, that have not yet been considered. Each possibility has locally the same Hamiltonian $-\frac{\hbar^2}{2}\Delta + V$, with the same potential V, and each possibility is equally well defined and equally reasonable. While in orthodox quantum mechanics it may seem more or less axiomatic that the configuration space $\mathcal{Q}$ is the space on which ψ_t is defined, $\mathcal{Q}$ appears in Bohmian mechanics also in another role: as the space in which Q_t moves. It is therefore less surprising from the Bohmian viewpoint, and easier to accept, that ψ_t is defined not on $\mathcal{Q}$ but on $\widehat{\mathcal{Q}}$. In the next section all wave functions will be complex-valued; in Section 4 we shall consider wave functions with higher-dimensional value spaces.

3 Scalar Wave Functions on the Covering Space

The motion of the configuration Q_t in $\mathcal{Q}$ is determined by a velocity vector field v_t on $\mathcal{Q}$, which may arise from a wave function ψ not on $\mathcal{Q}$ but instead on $\widehat{\mathcal{Q}}$, the universal covering space of $\mathcal{Q}$, in the following way: Suppose we are given a complex-valued map γ on the covering group $Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$, $\gamma : Cov(\widehat{\mathcal{Q}}, \mathcal{Q}) \to \mathbb{C}$, and suppose that a wave function $\psi : \widehat{\mathcal{Q}} \to \mathbb{C}$ satisfies the *periodicity condition associated with the topological factors* γ, i.e.,

$$\psi(\sigma\hat{q}) = \gamma_\sigma \psi(\hat{q}) \tag{9}$$

for every $\hat{q} \in \widehat{\mathcal{Q}}$ and $\sigma \in Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$. For (9) to be possible for a ψ that does not identically vanish, γ must be a representation of the covering group, as was first emphasized in [6]. To see this, let σ_1, $\sigma_2 \in Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$. Then we have the following equalities

$$\gamma_{\sigma_1\sigma_2}\psi(\hat{q}) = \psi(\sigma_1\sigma_2\hat{q}) = \gamma_{\sigma_1}\psi(\sigma_2\hat{q}) = \gamma_{\sigma_1}\gamma_{\sigma_2}\psi(\hat{q}). \tag{10}$$

We thus obtain the fundamental relation

$$\gamma_{\sigma_1\sigma_2} = \gamma_{\sigma_1}\gamma_{\sigma_2}, \tag{11}$$

establishing (since $\gamma_{\mathrm{Id}} = 1$) that γ is a representation.

Let $\pi_1(\mathcal{Q}, q)$ denote the *fundamental group of* $\mathcal{Q}$ *at a point* q and let π be the covering map (a local diffeomorphism) $\pi : \widehat{\mathcal{Q}} \to \mathcal{Q}$, also called the projection (the *covering fiber* for $q \in \mathcal{Q}$ is the set $\pi^{-1}(q)$ of points in $\widehat{\mathcal{Q}}$ that project to q under π). The 1-dimensional representations of the covering group are, via the canonical isomorphisms $\varphi_{\hat{q}} : Cov(\widehat{\mathcal{Q}}, \mathcal{Q}) \to \pi_1(\mathcal{Q}, q)$, $\hat{q} \in \pi^{-1}(q)$, in canonical correspondence with the 1-dimensional representations of any fundamental group $\pi_1(\mathcal{Q}, q)$: The different isomorphisms $\varphi_{\hat{q}}$, $\hat{q} \in \pi^{-1}(q)$, will transform a representation of $\pi_1(\mathcal{Q}, q)$ into representations of $Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$ that are conjugate. But the 1-dimensional representations are homomorphisms to the *abelian* multiplicative group of $\mathbb{C}$ and are thus invariant under conjugation.

From (9) it follows that $\nabla\psi(\sigma\hat{q}) = \gamma_\sigma \, \sigma^* \nabla\psi(\hat{q})$, where σ^* is the (push-forward) action of σ on tangent vectors, using that σ is an isometry. Thus, the velocity field $\hat{v}^\psi$ on $\widehat{\mathcal{Q}}$ associated with ψ according to

$$\hat{v}^\psi(\hat{q}) := \hbar \, \mathrm{Im} \, \frac{\nabla\psi}{\psi}(\hat{q}) \tag{12}$$

is projectable, i.e.,

$$\hat{v}^\psi(\sigma\hat{q}) = \sigma^* \hat{v}^\psi(\hat{q}), \tag{13}$$

and therefore gives rise to a velocity field v^ψ on $\mathcal{Q}$,

$$v^\psi(q) = \pi^* \, \hat{v}^\psi(\hat{q}) \tag{14}$$

where $\hat{q}$ is an arbitrary element of $\pi^{-1}(q)$.

If we let ψ evolve according to the Schrödinger equation on $\widehat{\mathcal{Q}}$,

$$i\hbar \frac{\partial\psi}{\partial t}(\hat{q}) = -\tfrac{\hbar^2}{2}\Delta\psi(\hat{q}) + \widehat{V}(\hat{q})\psi(\hat{q}) \tag{15}$$

with $\widehat{V}$ the lift of the potential V on $\mathcal{Q}$, then the periodicity condition (9) is preserved by the evolution, since, according to

$$i\hbar \frac{\partial\psi}{\partial t}(\sigma\hat{q}) \overset{(15)}{=} -\tfrac{\hbar^2}{2}\Delta\psi(\sigma\hat{q}) + \widehat{V}(\sigma\hat{q})\psi(\sigma\hat{q}) = -\tfrac{\hbar^2}{2}\Delta\psi(\sigma\hat{q}) + \widehat{V}(\hat{q})\psi(\sigma\hat{q}) \tag{16}$$

(note the different arguments in the potential), the functions $\psi \circ \sigma$ and $\gamma_\sigma \psi$ satisfy the same evolution equation (15) with, by (9), the same initial condition, and thus coincide at all times.

Therefore we can let the Bohmian configuration Q_t move according to v^{ψ_t},

$$\frac{dQ_t}{dt} = v^{\psi_t}(Q_t) = \hbar \, \pi^* \Big(\mathrm{Im} \, \frac{\nabla\psi}{\psi} \Big)(Q_t) = \hbar \, \pi^* \Big(\mathrm{Im} \, \frac{\nabla\psi}{\psi} \Big|_{\hat{q} \in \pi^{-1}(Q_t)} \Big). \tag{17}$$

One can also view the motion in this way: Given Q_0, choose $\widehat{Q}_0 \in \pi^{-1}(Q_0)$, let $\widehat{Q}_t$ move in $\widehat{\mathcal{Q}}$ according to $\hat{v}^{\psi_t}$, and set $Q_t = \pi(\widehat{Q}_t)$. Then the motion of Q_t is independent of the choice of $\widehat{Q}_0$ in the fiber over Q_0, and obeys (17).

If, as we shall assume from now on, $|\gamma_\sigma| = 1$ for all $\sigma \in Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$, i.e., if γ is a *unitary* representation (in $\mathbb{C}$) or a *character*, then the motion (17) also has an equivariant probability distribution, namely

$$\rho(q) = |\psi(\hat{q})|^2. \tag{18}$$

To see this, note that we have

$$|\psi(\sigma\hat{q})|^2 \overset{(9)}{=} |\gamma_\sigma|^2 |\psi(\hat{q})|^2 = |\psi(\hat{q})|^2, \tag{19}$$

so that the function $|\psi(\hat{q})|^2$ is projectable to a function on $\mathcal{Q}$ which we call $|\psi|^2(q)$ in this paragraph. From (15) we have that

$$\frac{\partial|\psi_t(\hat{q})|^2}{\partial t} = -\nabla\cdot\Big(|\psi_t(\hat{q})|^2\,\hat{v}^{\psi_t}(\hat{q})\Big)$$

and, by projection, that

$$\frac{\partial|\psi_t|^2(q)}{\partial t} = -\nabla\cdot\Big(|\psi_t|^2(q)\,v^{\psi_t}(q)\Big),$$

which coincides with the transport equation for a probability density ρ on $\mathcal{Q}$,

$$\frac{\partial\rho_t(q)}{\partial t} = -\nabla\cdot\Big(\rho_t(q)\,v^{\psi_t}(q)\Big).$$

Hence,

$$\rho_t(q) = |\psi_t|^2(q) \tag{20}$$

for all times if it is so initially; this is equivariance.

The relevant wave functions are those with

$$\int_{\mathcal{Q}} dq\,|\psi(\hat{q})|^2 = 1 \tag{21}$$

where the choice of $\hat{q}\in\pi^{-1}(q)$ is arbitrary by (19). The relevant Hilbert space, which we denote $L^2(\widehat{\mathcal{Q}},\gamma)$, thus consists of the measurable functions ψ on $\widehat{\mathcal{Q}}$ (modulo changes on null sets) satisfying (9) with

$$\int_{\mathcal{Q}} dq\,|\psi(\hat{q})|^2 < \infty. \tag{22}$$

It is a Hilbert space with the scalar product

$$\langle\phi,\psi\rangle = \int_{\mathcal{Q}} dq\,\overline{\phi(\hat{q})}\,\psi(\hat{q}). \tag{23}$$

Note that the value of the integrand at q is independent of the choice of $\hat{q}\in\pi^{-1}(q)$ since, by (9) and the fact that $|\gamma_\sigma| = 1$,

$$\overline{\phi(\sigma\hat{q})}\,\psi(\sigma\hat{q}) = \overline{\gamma_\sigma\,\phi(\hat{q})}\,\gamma_\sigma\,\psi(\hat{q}) = \overline{\phi(\hat{q})}\,\psi(\hat{q}).$$

We summarize the results of our reasoning.

Assertion 1. *Given a Riemannian manifold $\mathcal{Q}$ and a smooth function $V:\mathcal{Q}\to\mathbb{R}$, there is a Bohmian dynamics in $\mathcal{Q}$ with potential V for each character γ of the fundamental group $\pi_1(\mathcal{Q})$; it is defined by* (9), (15), *and* (17), *where the wave function ψ_t lies in $L^2(\widehat{\mathcal{Q}},\gamma)$ and has norm one.*

Assertion 1 provides as many dynamics as there are characters of $\pi_1(\mathcal{Q})$ because different characters $\gamma' \neq \gamma$ always define different dynamics. In particular, for the trivial character $\gamma_\sigma = 1$, we obtain the immediate dynamics, as defined by (2) and (1).

An important application of Assertion 1 is provided by identical particles without spin. The natural configuration space ${}^N\mathbb{R}^3$ for identical particles has fundamental group S_N, the group of permutations of N objects, which possesses two characters, the trivial character, $\gamma_\sigma = 1$, and the alternating character, $\gamma_\sigma = \mathrm{sgn}(\sigma) = 1$ or -1 depending on whether $\sigma \in S_N$ is an even or an odd permutation. The Bohmian dynamics associated with the trivial character is that of bosons, while the one associated with the alternating character is that of fermions. However, in a two-dimensional world there would be more possibilities since $\pi_1({}^N\mathbb{R}^2)$ is the braid group, whose generators $\sigma_i,\ i = 1, \ldots, N-1$, are a certain subset of braids that exchange two particles and satisfy the defining relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{for} \quad i \leq N-3, j \geq i+2,$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad \text{for} \quad i \leq N-2.$$

Thus, a character of the braid group assigns the same complex number $e^{i\beta}$ to each generator, and therefore, according to Assertion 1, each choice of β corresponds to a Bohmian dynamics; two-dimensional bosons correspond to $\beta = 0$ and two- dimensional fermions to $\beta = \pi$. The particles corresponding to the other possibilities are usually called *anyons*. They were first suggested in [15], and their investigation began in earnest with [11, 22]. See [16] for some more details and references.

4 Vector-Valued Wave Functions on the Covering Space

The analysis of Section 3 can be carried over with little change to the case of vector-valued wave functions, $\psi(q) \in W$. In this case, however, the topological factors may be given by any endomorphisms Γ_σ of W that form a representation of $Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$ and need not be restricted to characters, a possibility first mentioned in [20], Notes to Section 23.3. Rather than directly considering this case, we focus instead on one that is a bit more general and that will require a new sort of topological factor, that of wave functions that are sections of a vector bundle. The topological factors for this case will be expressed as *periodicity sections*, i.e., parallel unitary sections of the endomorphism bundle indexed by the covering group and satisfying a certain composition law, or, equivalently, as *holonomy-twisted representations* of $\pi_1(\mathcal{Q})$.

If E is a vector bundle over $\mathcal{Q}$, then the lift of E, denoted by $\widehat{E}$, is a vector bundle over $\widehat{\mathcal{Q}}$; the fiber space at $\hat{q}$ is defined to be the fiber space of E at q, $\widehat{E}_{\hat{q}} := E_q$, where $q = \pi(\hat{q})$. It is important to realize that with this construction, it makes sense to ask whether $v \in \widehat{E}_{\hat{q}}$ is equal to $w \in \widehat{E}_{\hat{r}}$ whenever $\hat{q}$ and $\hat{r}$ are elements of the same covering fiber. Equivalently, $\widehat{E}$ is the pull-back of E through $\pi : \widehat{\mathcal{Q}} \to \mathcal{Q}$. As a particular example, the lift of the tangent bundle of $\mathcal{Q}$ to $\widehat{\mathcal{Q}}$ is canonically isomorphic to the tangent bundle of $\widehat{\mathcal{Q}}$. Sections of E or $E \otimes E^*$ can be lifted to sections of $\widehat{E}$ respectively $\widehat{E} \otimes \widehat{E}^*$.

If E is a Hermitian vector bundle, then so is $\widehat{E}$. The wave function ψ that we consider here is a section of $\widehat{E}$, so that $\psi(\hat{q})$ is a vector in the $\hat{q}$-dependent Hermitian vector space $\widehat{E}_{\hat{q}}$. V is a section of the bundle $E \otimes E^*$, i.e., $V(q)$ is an element of $E_q \otimes E_q^*$. To indicate that every $V(q)$ is a Hermitian endomorphism of E_q, we say that V is a Hermitian section of $E \otimes E^*$.

Since $\psi(\sigma\hat{q})$ and $\psi(\hat{q})$ lie in the same space $E_q = \widehat{E}_{\hat{q}} = \widehat{E}_{\sigma\hat{q}}$, a periodicity condition can be of the form

$$\psi(\sigma\hat{q}) = \Gamma_\sigma(\hat{q})\,\psi(\hat{q}) \tag{24}$$

for $\sigma \in Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$, where $\Gamma_\sigma(\hat{q})$ is an endomorphism $E_q \to E_q$. By the same argument as in (10), the condition for (24) to be possible, if $\psi(\hat{q})$ can be any element of $\widehat{E}_{\hat{q}}$, is the composition law

$$\Gamma_{\sigma_1\sigma_2}(\hat{q}) = \Gamma_{\sigma_1}(\sigma_2\hat{q})\,\Gamma_{\sigma_2}(\hat{q}). \tag{25}$$

Note that this law differs from the one $\Gamma(\hat{q})$ would satisfy if it were a representation, which reads $\Gamma_{\sigma_1\sigma_2}(\hat{q}) = \Gamma_{\sigma_1}(\hat{q})\,\Gamma_{\sigma_2}(\hat{q})$, since in general $\Gamma(\sigma\hat{q})$ need not be the same as $\Gamma(\hat{q})$.

For periodicity (24) to be preserved under the Schrödinger evolution,

$$i\hbar\frac{\partial\psi}{\partial t}(\hat{q}) = -\tfrac{\hbar^2}{2}\Delta\psi(\hat{q}) + \widehat{V}(\hat{q})\,\psi(\hat{q}), \tag{26}$$

we need that multiplication by $\Gamma_\sigma(\hat{q})$ commute with the Hamiltonian. Observe that

$$[H,\Gamma_\sigma]\psi(\hat{q}) = -\tfrac{\hbar^2}{2}(\Delta\Gamma_\sigma(\hat{q}))\psi(\hat{q}) - \hbar^2(\nabla\Gamma_\sigma(\hat{q}))\cdot(\nabla\psi(\hat{q})) + [\widehat{V}(\hat{q}),\Gamma_\sigma(\hat{q})]\,\psi(\hat{q}). \tag{27}$$

Since we can choose ψ such that, for any one particular $\hat{q}$, $\psi(\hat{q}) = 0$ and $\nabla\psi(\hat{q})$ is any element of $\mathbb{C}T_{\hat{q}}\widehat{\mathcal{Q}} \otimes E_q$ we like, we must have that

$$\nabla\Gamma_\sigma(\hat{q}) = 0 \tag{28}$$

for all $\sigma \in Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$ and all $\hat{q} \in \widehat{\mathcal{Q}}$, i.e., that Γ_σ is parallel. Inserting this in (27), the first two terms on the right hand side vanish. Since we can choose for $\psi(\hat{q})$ any element of E_q we like, we must have that

$$[\widehat{V}(\hat{q}),\Gamma_\sigma(\hat{q})] = 0 \tag{29}$$

for all $\sigma \in Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$ and all $\hat{q} \in \widehat{\mathcal{Q}}$. Conversely, assuming (28) and (29), we obtain that Γ_σ commutes with H for every $\sigma \in Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$, so that the periodicity (24) is preserved.

From (24) and (28) it follows that $\nabla\psi(\sigma\hat{q}) = (\sigma^* \otimes \Gamma_\sigma(\hat{q}))\nabla\psi(\hat{q})$. If every $\Gamma_\sigma(\hat{q})$ is *unitary*, as we assume from now on, the velocity field $\hat{v}^\psi$ on $\widehat{\mathcal{Q}}$ associated with ψ according to

$$\hat{v}^\psi(\hat{q}) := \hbar\,\mathrm{Im}\,\frac{(\psi,\nabla\psi)}{(\psi,\psi)}(\hat{q}) \tag{30}$$

is projectable, $\hat{v}^\psi(\sigma\hat{q}) = \sigma^*\hat{v}^\psi(\hat{q})$, and gives rise to a velocity field v^ψ on $\mathcal{Q}$. We let the configuration move according to v^{ψ_t},

$$\frac{dQ_t}{dt} = v^{\psi_t}(Q_t) = \hbar\,\pi^*\Big(\mathrm{Im}\,\frac{(\psi,\nabla\psi)}{(\psi,\psi)}\Big)(Q_t). \tag{31}$$

Definition 2. *Let E be a Hermitian bundle over the manifold $\mathcal{Q}$. A* periodicity section Γ *over E is a family indexed by $Cov(\widehat{\mathcal{Q}},\mathcal{Q})$ of unitary parallel sections Γ_σ of $\widehat{E}\otimes\widehat{E}^*$ satisfying the composition law* (25)*.*

Since $\Gamma_\sigma(\hat{q})$ is unitary, one sees as before that the probability distribution

$$\rho(q) = (\psi(\hat{q}),\psi(\hat{q})) \tag{32}$$

does not depend on the choice of $\hat{q}\in\pi^{-1}(q)$ and is equivariant.

As usual, we define for any periodicity section Γ the Hilbert space $L^2(\widehat{\mathcal{Q}},\widehat{E},\Gamma)$ to be the set of measurable sections ψ of $\widehat{E}$ (modulo changes on null sets) satisfying (24) with

$$\int_{\mathcal{Q}} dq\,(\psi(\hat{q}),\psi(\hat{q})) < \infty, \tag{33}$$

endowed with the scalar product

$$\langle\phi,\psi\rangle = \int_{\mathcal{Q}} dq\,(\phi(\hat{q}),\psi(\hat{q})). \tag{34}$$

As before, the value of the integrand at q is independent of the choice of $\hat{q}\in\pi^{-1}(q)$.

We summarize the results of our reasoning.

Assertion 2. *Given a Hermitian bundle E over the Riemannian manifold $\mathcal{Q}$ and a Hermitian section V of $E\otimes E^*$, there is a Bohmian dynamics for each periodicity section Γ commuting (pointwise) with $\widehat{V}$ (cf. (29)); it is defined by* (24)*,* (26)*, and* (31)*, where the wave function ψ_t lies in $L^2(\widehat{\mathcal{Q}},\widehat{E},\Gamma)$ and has norm 1.*

Every character γ of $Cov(\widehat{\mathcal{Q}},\mathcal{Q})$ (or of $\pi_1(\mathcal{Q})$) defines a periodicity section by setting

$$\Gamma_\sigma(\hat{q}) := \gamma_\sigma \mathrm{Id}_{\widehat{E}_{\hat{q}}}. \tag{35}$$

It commutes with every potential V. Conversely, a periodicity section Γ that commutes with every potential must be such that every $\Gamma_\sigma(\hat{q})$ is a multiple of the identity, $\Gamma_\sigma(\hat{q}) = \gamma_\sigma(\hat{q})\,\mathrm{Id}_{\widehat{E}_{\hat{q}}}$. By unitarity, $|\gamma_\sigma| = 1$; by parallelity (28), $\gamma_\sigma(\hat{q}) = \gamma_\sigma$ must be constant; by the composition law (25), γ must be a homomorphism, and thus a character.

We briefly indicate how a periodicity section Γ corresponds to something like a representation of $\pi_1(\mathcal{Q})$. Fix a $\hat{q}\in\widehat{\mathcal{Q}}$. Then $Cov(\widehat{\mathcal{Q}},\mathcal{Q})$ can be identified with $\pi_1(\mathcal{Q}) = \pi_1(\mathcal{Q},\pi(\hat{q}))$ via $\varphi_{\hat{q}}$. Since the sections Γ_σ of $\widehat{E}\otimes\widehat{E}^*$ are parallel, $\Gamma_\sigma(\hat{r})$ is determined for every $\hat{r}$ by $\Gamma_\sigma(\hat{q})$. (Note in particular that the parallel transport $\Gamma_\sigma(\tau\hat{q})$ of $\Gamma_\sigma(\hat{q})$ from $\hat{q}$ to $\tau\hat{q}, \tau\in Cov(\widehat{\mathcal{Q}},\mathcal{Q})$, may differ from $\Gamma_\sigma(\hat{q})$.) Thus, the periodicity section Γ is completely determined by the endomorphisms $\Gamma_\sigma := \Gamma_\sigma(\hat{q})$ of E_q, $\sigma \in Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$, which satisfy the composition law

$$\Gamma_{\sigma_1\sigma_2} = h_{\alpha_2}\Gamma_{\sigma_1}h_{\alpha_2}^{-1}\Gamma_{\sigma_2}\,, \tag{36}$$

where α_2 is any loop in $\mathcal{Q}$ based at $\pi(\hat{q})$ whose lift starting at $\hat{q}$ leads to $\sigma_2\hat{q}$, and h_{α_2} is the associated holonomy endomorphism of E_q. Since (36) is not the composition law $\Gamma_{\sigma_1\sigma_2} = \Gamma_{\sigma_1}\Gamma_{\sigma_2}$ of a representation, the Γ_σ form, not a representation of $\pi_1(\mathcal{Q})$, but what we call a *holonomy-twisted representation*.

The situation where the wave function assumes values in a fixed Hermitian space W, instead of a bundle, corresponds to the trivial Hermitian bundle $E = \mathcal{Q} \times W$ (i.e., with the trivial connection, for which parallel transport is the identity on W). Then, parallelity (28) implies that $\Gamma_\sigma(\hat{r}) = \Gamma_\sigma(\hat{q})$ for any $\hat{r}, \hat{q} \in \widehat{\mathcal{Q}}$, or $\Gamma_\sigma(\hat{q}) = \Gamma_\sigma$, so that (25) becomes the usual composition law $\Gamma_{\sigma_1\sigma_2} = \Gamma_{\sigma_1}\Gamma_{\sigma_2}$ and Γ is a unitary representation of $Cov(\widehat{\mathcal{Q}}, \mathcal{Q})$.

The most important case of topological factors that are characters is provided by identical particles *with spin*. In fact, for this case, Assertion 2 entails the same conclusions we arrived at the end of Section 3, even for particles with spin. To understand how this comes about, consider the potential occurring in the Pauli equation for N identical particles with spin,

$$V(q) = -\mu \sum_{\boldsymbol{q}\in q} \boldsymbol{B}(\boldsymbol{q}) \cdot \boldsymbol{\sigma}_{\boldsymbol{q}} \tag{37}$$

on the spin bundle (8) over ${}^N\mathbb{R}^3$, with $\boldsymbol{\sigma}_{\boldsymbol{q}}$ the vector of spin matrices acting on the spin space of the particle at $\boldsymbol{q}$. Clearly, the algebra generated by $\{V(q)\}$ arising from all possible choices of the magnetic field $\boldsymbol{B}$ is $\mathrm{End}(E_q)$. Thus the only holonomy-twisted representations that define a dynamics for all magnetic fields are those given by a character.[5]

An example of a topological factor that is not a character is provided by the Aharonov–Casher variant [1] of the Aharonov–Bohm effect, according to which a neutral spin-1/2 particle that carries a magnetic moment μ acquires a nontrivial phase while encircling a charged wire $\mathcal{C}$. A way of understanding how this effect comes about is in terms of the non-relativistic Hamiltonian $-\frac{\hbar^2}{2}\Delta + V$ based on a nontrivial connection $\nabla = \nabla_{\text{trivial}} - \frac{i\mu}{\hbar}\boldsymbol{E} \times \boldsymbol{\sigma}$ on the vector bundle $\mathbb{R}^3 \times \mathbb{C}^2$. Suppose the charge density $\varrho(\boldsymbol{q})$ is invariant under translations in the direction $\boldsymbol{e} \in \mathbb{R}^3$, $\boldsymbol{e}^2 = 1$ in which the wire is oriented. Then the charge per unit length λ is given by the integral

$$\lambda = \int_D \varrho(\boldsymbol{q})\, dA \tag{38}$$

over the cross-section disk D in any plane perpendicular to $\boldsymbol{e}$. The restriction of this

[5] In fact, it can be shown [7] that the only holonomy-twisted representations for a magnetic field $\boldsymbol{B}$ that is not parallel must be a character.

connection, outside of $\mathcal{C}$, to any plane Σ orthogonal to the wire turns out to be flat[6] so that its restriction to the intersection $\mathcal{Q}$ of $\mathbb{R}^3 \setminus \mathcal{C}$ with the orthogonal plane can be replaced, as in the Aharonov–Bohm case, by the trivial connection if we introduce a periodicity condition on the wave function with the topological factor

$$\Gamma_1 = \exp\Big(-\frac{4\pi i\mu\lambda}{\hbar}\,\boldsymbol{e}\cdot\boldsymbol{\sigma}\Big). \tag{39}$$

In this way we obtain a representation $\Gamma : \pi_1(\mathcal{Q}) \to SU(2)$ that is not given by a character.

Another example of a topological factor that is not a character and which can be generalized to a nonabelian representation is provided by a higher-dimensional version of the Aharonov–Bohm effect: one may replace the vector potential in the Aharonov–Bohm setting by a non-abelian gauge field (à la Yang–Mills) whose field strength (curvature) vanishes outside a cylinder $\mathcal{C}$ but not inside; the value space W (now corresponding not to spin but to, say, quark color) has dimension greater than one, and the difference between two wave packets that have passed $\mathcal{C}$ on different sides is given in general, not by a phase, but by a unitary endomorphism Γ of W. In this example, involving one cylinder, the representation Γ, though given by matrices that are not multiples of the identity, is nonetheless abelian, since $\pi_1(\mathcal{Q}) \cong \mathbb{Z}$ is an abelian group. However, when two or more cylinders are considered, we obtain a non-abelian representation Γ, since when $\mathcal{Q}$ is $\mathbb{R}^3$ minus two disjoint solid cylinders its fundamental group is isomorphic to the non-abelian group $\mathbb{Z} * \mathbb{Z}$, where $*$ denotes the free product of groups, generated by loops σ_1 and σ_2 surrounding one or the other of the cylinders. One can easily arrange that the matrices Γ_{σ_i} corresponding to loops σ_i, $i = 1, 2$, fail to commute, so that Γ is nonabelian.

Our last example involves a holonomy-twisted representation Γ that is not a representation in the ordinary sense. Consider N fermions, each as in the previous examples, moving in $M = \mathbb{R}^3 \setminus \cup_i \mathcal{C}_i$, where $\mathcal{C}_i$ are one or more disjoint solid cylinders. More generally, consider N fermions, each having 3-dimensional configuration space M and value space W (which may incorporate spin or "color" or both). Then the configuration space $\mathcal{Q}$ for the N fermions is the set ${}^N M$ of all N-element subsets of M, with universal covering space $\widehat{\mathcal{Q}} = \widehat{{}^N M} = \widehat{M}^N \setminus \Delta$ with Δ the extended diagonal, the set of points in $\widehat{M}^N$ whose projection to M^N lies in its coincidence set. Every diffeomorphism $\sigma \in Cov(\widehat{{}^N M}, {}^N M)$ can be expressed as a product

$$\sigma = p\tilde{\sigma} \tag{40}$$

where $p \in S_N$ and $\tilde{\sigma} = (\sigma^{(1)}, \ldots, \sigma^{(N)}) \in Cov(\widehat{M}, M)^N$ and these act on $\hat{q} = (\hat{\boldsymbol{q}}_1, \ldots, \hat{\boldsymbol{q}}_N) \in \widehat{M}^N$ as follows:

$$\tilde{\sigma}\hat{q} = (\sigma^{(1)}\hat{\boldsymbol{q}}_1, \ldots, \sigma^{(N)}\hat{\boldsymbol{q}}_N) \tag{41}$$

[6]The curvature is $\Omega = d_{\text{trivial}}\boldsymbol{\omega} + \boldsymbol{\omega}\wedge\boldsymbol{\omega}$ with $\boldsymbol{\omega} = -i\frac{\mu}{\hbar}\boldsymbol{E}\times\boldsymbol{\sigma}$. The 2-form Ω is dual to the vector $\nabla_{\text{trivial}}\times\boldsymbol{\omega} + \boldsymbol{\omega}\times\boldsymbol{\omega} = i\frac{\mu}{\hbar}(\nabla\cdot\boldsymbol{E})\boldsymbol{\sigma} - i\frac{\mu}{\hbar}(\boldsymbol{\sigma}\cdot\nabla)\boldsymbol{E} - 2i(\frac{\mu}{\hbar})^2(\boldsymbol{\sigma}\cdot\boldsymbol{E})\boldsymbol{E}$. Outside the wire, the first term vanishes and, noting that $\boldsymbol{E}\cdot\boldsymbol{e} = 0$, the other two terms have vanishing component in the direction of $\boldsymbol{e}$ and thus vanish when integrated over any region within an orthogonal plane.

and

$$p\hat{q} = (\hat{\boldsymbol{q}}_{p^{-1}(1)}, \ldots, \hat{\boldsymbol{q}}_{p^{-1}(N)}). \tag{42}$$

Thus

$$\sigma\hat{q} = (\sigma^{(p^{-1}(1))}\hat{\boldsymbol{q}}_{p^{-1}(1)}, \ldots, \sigma^{(p^{-1}(N))}\hat{\boldsymbol{q}}_{p^{-1}(N)}). \tag{43}$$

Moreover, the representation (40) of σ is unique. Thus, since

$$\sigma_1\sigma_2 = p_1\tilde{\sigma}_1 p_2\tilde{\sigma}_2 = (p_1p_2)(p_2^{-1}\tilde{\sigma}_1 p_2\tilde{\sigma}_2) \tag{44}$$

with $p_2^{-1}\tilde{\sigma}_1 p_2 = (\sigma_1^{(p_2(1))}, \ldots, \sigma_1^{(p_2(N))}) \in Cov(\widehat{M}, M)^N$, we find that $Cov(\widehat{{}^N M}, {}^N M)$ is a semidirect product of S_N and $Cov(\widehat{M}, M)^N$, with product given by

$$\sigma_1\sigma_2 = (p_1, \tilde{\sigma}_1)(p_2, \tilde{\sigma}_2) = (p_1p_2, p_2^{-1}\tilde{\sigma}_1 p_2\tilde{\sigma}_2). \tag{45}$$

Wave functions for the N fermions are sections of the lift $\widehat{E}$ to $\widehat{\mathcal{Q}}$ of the bundle E over $\mathcal{Q}$ with fiber

$$E_q = \bigotimes_{\boldsymbol{q}\in q} W \tag{46}$$

and (nontrivial) connection inherited from the trivial connection on $M \times W$. If the dynamics for $N = 1$ involves wave functions on $\widehat{M}$ obeying (24) with topological factor $\Gamma_\sigma(\hat{\boldsymbol{q}}) = \Gamma_\sigma$ given by a unitary representation of $\pi_1(M)$ (i.e., independent of $\hat{\boldsymbol{q}}$), then the N fermion wave function obeys (24) with topological factor

$$\Gamma_\sigma(\hat{q}) = \operatorname{sgn}(p) \bigotimes_{\boldsymbol{q}\in\pi(\hat{q})} \Gamma_{\sigma^{(i_{\hat{q}}(\boldsymbol{q}))}} \equiv \operatorname{sgn}(p)\Gamma_{\tilde{\sigma}}(\hat{q}) \tag{47}$$

where for $\hat{q} = (\hat{\boldsymbol{q}}_1, \ldots, \hat{\boldsymbol{q}}_N)$, $\pi(\hat{q}) = \{\pi_M(\hat{\boldsymbol{q}}_1), \ldots, \pi_M(\hat{\boldsymbol{q}}_N)\}$ and $i_{\hat{q}}(\pi_M(\hat{\boldsymbol{q}}_j)) = j$. Since

$$\Gamma_{\tilde{\sigma}_1\tilde{\sigma}_2}(\hat{q}) = \Gamma_{\tilde{\sigma}_1}(\hat{q})\,\Gamma_{\tilde{\sigma}_2}(\hat{q}) \tag{48}$$

we find, using (45) and (48), that

$$\Gamma_{\sigma_1\sigma_2}(\hat{q}) = \operatorname{sgn}(p_1p_2)\Gamma_{p_2^{-1}\tilde{\sigma}_1 p_2\tilde{\sigma}_2}(\hat{q}) \tag{49a}$$

$$= \operatorname{sgn}(p_1)\Gamma_{p_2^{-1}\tilde{\sigma}_1 p_2}(\hat{q})\operatorname{sgn}(p_2)\Gamma_{\tilde{\sigma}_2}(\hat{q}) \tag{49b}$$

$$= P_2\Gamma_{\sigma_1}(\hat{q})P_2^{-1}\Gamma_{\sigma_2}(\hat{q}), \tag{49c}$$

which agrees with (36) since the holonomy on the bundle E is given by permutations P acting on the tensor product (46).

5 Conclusions

We have investigated the possible quantum theories on a topologically nontrivial configuration space $\mathcal{Q}$ from the point of view of Bohmian mechanics, which is fundamentally concerned with the motion of matter in physical space, represented by the evolution of a point in configuration space.

Our goal was to find all Bohmian dynamics in $\mathcal{Q}$, where the wave functions may be sections of a Hermitian vector bundle E. What "all" Bohmian dynamics means is not obvious; we have followed one approach to what it can mean; other approaches will be described in future works. The present approach uses wave functions ψ that are defined on the universal covering space $\widehat{\mathcal{Q}}$ of $\mathcal{Q}$ and satisfy a periodicity condition ensuring that the Bohmian velocity vector field on $\widehat{\mathcal{Q}}$ defined in terms of ψ can be projected to $\mathcal{Q}$. We have arrived in this way at a natural class of Bohmian dynamics beyond the immediate Bohmian dynamics. Such a dynamics is defined by a potential and some information encoded in "topological factors," which form either a character (one-dimensional unitary representation) of the fundamental group of the configuration space, $\pi_1(\mathcal{Q})$, or a more general algebraic-geometrical object, a holonomy-twisted representation Γ. Only those dynamics associated with characters are compatible with *every* potential, as one would desire for what could be considered a version of quantum mechanics in $\mathcal{Q}$. We have thus arrived at the known fact that for every character of $\pi_1(\mathcal{Q})$ there is a version of quantum mechanics in $\mathcal{Q}$. A consequence, which will be discussed in detail in a sister paper [7], is the symmetrization postulate for identical particles. These different quantum theories emerge naturally when one contemplates the possibilities for defining a Bohmian dynamics in $\mathcal{Q}$.

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